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Optimal Feedback Control of a Nonlinear System: Wing Rock Example

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A procedure is presented for optimizing a state feedback control law for a nonlinear system with respect to a positive performance index. The Hamilton–Jacobi–Bellman equation is employed to derive the optimality equations wherein this performance index is minimized. The closed-loop Lyapunov function is assumed to have the same matrix form of state variables as the performance index. The constant interpolated terms of these matrix forms are easily determined so as to guarantee their positive definitenesses. The optimal nonlinear system is asymptotically stable in the large, as both the closed-loop Lyapunov function and performance index are positive definite. An unstable wing rock equation of motion is employed to illustrate this method. It is shown that the wing rock model using nonlinear state feedback is asymptotically stable in the large. Both optimal linear and nonlinear state feedback cases are evaluated.

Nomenclature

A	= linear system state matrix
a_i	= coefficients of what dependent on the angle of attack
B	= control matrix
b_1, b_2	= coefficients of the wing rock equation
c_1, c_2	= constant values
$f(x)$	= nonlinear terms of the system
H	= Hamiltonian equation
J	= performance index, also called cost function
n	= positive integer relative to the highest order term of $f(x)$
P	= Lyapunov matrix
$P_{i,j}$	= submatrix of P
Q	= interpolated term of $q(x)$
$Q_{i,j}$	= submatrix of Q
$q(x)$	= nonlinear state performance index
R	= control weight matrix
u	= control inputs
$V(x)$	= Lyapunov function
x	= system state variables
α	= angle of attack
ε	= small positive number
μ_1, μ_2	= coefficients of the wing rock equation
ϕ	= roll angle
$\dot{\phi}$	= roll rate
ϖ	= wing rock limit cycle frequency

I. Introduction

THE many aspects of the optimal control problem have been studied since the Hamilton–Jacobi–Bellman (HJB) equation was derived. Early applications have been in linear quadratic regulator (LQR) problems. LQR is guaranteed to have an optimal and asymptotically stable solution because the closed-loop Lyapunov function is positive definite or positive semidefinite and its time derivative is negative definite.

Consider a nonlinear system as follows:

$$\dot{x} = Ax + f(x) + Bu$$

and the performance index

$$J = \frac{1}{2}\int_0^\infty [q(x) + u^T Ru]\,dt$$

For the extension of the LQR problem to the nonlinear optimal control problem, the following conditions must be satisfied.

Condition 1:

$$V(x) \geq 0, \qquad V(0) = 0$$

Condition 2:

$\min\{H\}$

$$= \min\left\{\frac{1}{2}q(x) + \frac{1}{2}u^T Ru + \frac{\partial V(x)}{\partial x}[Ax + f(x) + Bu]\right\} = 0$$

Condition 3:

$$q(x) \geq 0$$

Condition 2 is called the HJB equation that gives the system

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = -\frac{1}{2}[q(x) + u^T Ru] \leq 0$$

if condition 3 is satisfied.

One of the most popular optimal control methods used in nonlinear systems is the method of singular perturbations.[1–4] This method assumes that the closed-loop Lyapunov function is a series expansion of a small parameter. Through evaluation of the HJB equation with this Lyapunov function, the optimal control is determined. This method, however, does not provide a way to check whether or not the Lyapunov function is positive definite; that is, this method does not offer a way to ensure that the system is asymptotically stable in the large. Because of this uncertainty regarding the closed-loop Lyapunov function, Garrard[1] stated that "The control created by a small perturbation [singular perturbation] is useful only within some domain of asymptotical stability surrounding the origin." Furthermore, the designer must carefully weigh the cost required for the control to create the better behavior.

In fact, no matter whether or not the linear system or the nonlinear system uses nonlinear feedback control, the system's closed-loop Lyapunov function has to be positive definite. This point, however, is still neglected in the current literature in this area, such as in Refs. 5–9. Some works[10,11] mention the importance of the listed three conditions. Goh[10] used an appropriately trained feedforward neural network as a nonlinear controller on a nonlinear system, where the HJB equation and a positive definiteness of the closed-loop Lyapunov function were satisfied. The problem, however, was restricted to the neural network's training domain. Bernstein[11] indicated that the Lyapunov function must be positive definite, and so he used a fourth-order positive definite Lyapunov function to calculate

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the HJB equation and find the optimal control. The method, however, was limited to fourth-order systems, and the flexibility of the performance index was also limited.

As with the LQR problem, the performance index plays a very important role in the problem of optimal control of nonlinear systems. When a positive definite Lyapunov function cannot be found, this purpose usually can be achieved through changing the cost function. Leeper and Mulholland[12] used an appropriate and reasonable performance index such that the system was asymptotically stable, since the nonlinear term of the state equation could be canceled.

In the current paper, a procedure is presented that produces a closed-loop Lyapunov function and a performance index that are positive definite, and the HJB equation is also satisfied. This means that the system is asymptotically stable in the large.

II. Method of Analysis

A. Problem Statement and Methodology

Consider a time-invariant nonlinear system

$$\dot{x} = Ax + f(x) + Bu \tag{1}$$

where A, B, x, and u have proper dimensions, and $f(x)$ represents the higher-order term with respect to the state variable x. Therefore, $f(x)$ may be written in the following form:

$$f(x) = f_2(x) + f_3(x) + \cdots + f_{2n-1}(x); \qquad n \geq 2 \tag{2}$$

Note that subscripts of $f(x)$, the performance index, the closed-loop Lyapunov function, and the Hamiltonian matrix represent the order of the state variable x. One can define a performance index

$$J = \frac{1}{2}\int_0^\infty [q(x) + u^T R u]\, dt \tag{3}$$

where

$$q(x) = \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}^T \begin{bmatrix} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,n} \\ Q_{2,1} & P_{2,2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ Q_{n,1} & \cdots & \cdots & Q_{n,n} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \tag{4}$$

with $q(x) \geq 0$, and $Q_{i,j} = Q_{j,i}$ when $i \neq j$. Also, R is assumed to be symmetric, orthogonal, and positive definite. Therefore,

$$q(x) = q_2(x) + q_3(x) + q_4(x) + \cdots + q_{2n}(x)$$

where

$$q_2(x) = \tfrac{1}{2}x^T Q_{1,1} x$$

$$q_3(x) = \tfrac{1}{2}x^T Q_{1,2} x^2 + \tfrac{1}{2}(x^2)^T Q_{2,1} x$$

$$q_4(x) = \tfrac{1}{2}x^T Q_{1,3} x^3 + \tfrac{1}{2}(x^2)^T Q_{2,2} x^2 + \tfrac{1}{2}(x^3)^T Q_{3,1} x^1$$

$$\vdots \qquad \vdots \qquad \vdots$$

and

$$q_{2n}(x) = \tfrac{1}{2}(x^n)^T Q_{n,n} x^n$$

The optimal control problem of the system in Eq. (1) is to find the state feedback gain

$$u = u(x)$$

such that the performance index (3) is minimized. The HJB equation shown as follows is applied to find the optimal solution of the system,

$$H = \frac{1}{2}q(x) + \frac{1}{2}u^T R u + \frac{\partial V(x)}{\partial x}[Ax + f(x) + Bu] = 0 \tag{5}$$

where

$$\frac{\partial V(x)}{\partial x} = \left[\frac{\partial V(x)}{\partial x_1}, \frac{\partial V(x)}{\partial x_2}, \ldots, \frac{\partial V(x)}{\partial x_n}\right] \tag{6}$$

The optimal control is

$$u(x) = -R^{-1}B^T\left[\frac{\partial V(x)}{\partial x}\right]^T \tag{7}$$

when

$$\frac{\partial H}{\partial x} = 0$$

Substituting Eq. (7) into Eq. (5) results in

$$H = \frac{1}{2}q(x) - \frac{1}{2}\frac{\partial V(x)}{\partial x}BRB^T\left[\frac{\partial V(x)}{\partial x}\right]^T + \frac{\partial V(x)}{\partial x}[Ax + f(x)] = 0 \tag{8}$$

Assume the Lyapunov function has the same form as the cost function,

$$V(x) = \frac{1}{2}\begin{bmatrix} x \\ x^2 \\ \vdots \\ x^n \end{bmatrix}^T \begin{bmatrix} P_{1,1} & P_{1,2} & \cdots & P_{1,n} \\ P_{2,1} & P_{2,2} & & \vdots \\ \vdots & & \ddots & \vdots \\ P_{n,1} & \cdots & \cdots & P_{n,n} \end{bmatrix} \begin{bmatrix} x \\ x^2 \\ \vdots \\ x^n \end{bmatrix} \tag{9}$$

and $V(x) \geq 0$, where $P_{i,j} = P_{j,i}$ when $i \neq j$. Therefore,

$$V(x) = V_2(x) + V_3(x) + V_4(x) + \cdots + V_{2n}(x)$$

where

$$V_2(x) = \tfrac{1}{2}x^T P_{1,1} x$$

$$V_3(x) = \tfrac{1}{2}x^T P_{1,2} x^2 + \tfrac{1}{2}(x^2)^T P_{2,1} x$$

$$\vdots \qquad \vdots \qquad \vdots$$

and

$$V_{2n}(x) = \tfrac{1}{2}(x^n)^T P_{n,n} x^n$$

Note that the performance index and the Lyapunov function are symmetric.

Let $H = H_2 + H_3 + H_4 + \cdots + H_{2n}$. Substituting Eqs. (9), (4), and (2) into Eq. (8) and rearranging based on the order of the state variable,

$$H_2 = \frac{1}{2}q_2(x) - \frac{1}{2}\frac{\partial V_2(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_2(x)}{\partial x}\right]^T + \frac{\partial V_2(x)}{\partial x}Ax = 0 \tag{10}$$

$$H_3 = \frac{1}{2}q_3(x) - \frac{1}{2}\frac{\partial V_2(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_3(x)}{\partial x}\right]^T - \frac{1}{2}\frac{\partial V_3(x)}{\partial x} \times BR^{-1}B^T\left[\frac{\partial V_2(x)}{\partial x}\right]^T + \frac{\partial V_3(x)}{\partial x}Ax + \frac{\partial V_2(x)}{\partial x}f_2(x) = 0 \tag{11}$$

$$H_4 = \frac{1}{2}q_4(x) - \frac{1}{2}\frac{\partial V_2(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_4(x)}{\partial x}\right]^T - \frac{1}{2}\frac{\partial V_3(x)}{\partial x} \times BR^{-1}B^T\left[\frac{\partial V_3(x)}{\partial x}\right]^T - \frac{1}{2}\frac{\partial V_4(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_2(x)}{\partial x}\right]^T + \frac{\partial V_4(x)}{\partial x}Ax + \frac{\partial V_3(x)}{\partial x}f_2(x) + \frac{\partial V_2(x)}{\partial x}f_3(x) = 0 \tag{12}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$H_{2n} = \frac{1}{2}q_{2n}(x) - \frac{1}{2}\frac{\partial V_2(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_{2n}(x)}{\partial x}\right]^T - \cdots - \frac{1}{2}\frac{\partial V_{2n}(x)}{\partial x}BR^{-1}B^T\left[\frac{\partial V_2(x)}{\partial x}\right]^T + \frac{\partial V_{2n}(x)}{\partial x}Ax + \frac{\partial V_{2n-1}(x)}{\partial x}f_2(x) + \cdots + \frac{\partial V_2(x)}{\partial x}f_{2n-1}(x) = 0 \tag{13}$$

Table 1 Coefficients for the wing rock motion from Ref. 13

α	c_1	c_2	a_1	a_2	a_3	a_4	a_5
21.5	0.354	0.001	−0.04207	0.01456	0.04714	−0.18583	0.24234
22.5	0.354	0.001	−0.04681	0.01966	0.05671	−0.22691	0.59065
25	0.354	0.001	−0.05686	0.03254	0.07334	−0.3597	1.4681

Solving Eq. (10), the algebraic Riccati equation is obtained,

$$A^T P_{1,1} + P_{1,1} A + Q_{1,1} - P_{1,1} B R^{-1} B^T P_{1,1} = 0 \qquad (14)$$

Solving the algebraic Riccati equation, and substituting $P_{1,1}$ back into Eq. (11), $P_{2,1}$ can be found. The remaining values of P can be found by successive substitution of the lower order Lyapunov parameters into the higher order Lyapunov parameters until $P_{n,n}$ is found. After all Lyapunov parameters have been determined, stability of the Lyapunov function has to be checked: if $V(x) > 0$ and $q(x) > 0$, then the nonlinear optimal control is asymptotically stable in the large.

B. Nonlinear Optimal Control Procedure

The general procedure for solving the Lyapunov function $V(x)$ is as follows.

Step 1: Find the highest order $(2n - 1)$ of the state variable from the state equation, so that n can be properly selected.

Step 2: Assume $q(x)$ and $V(x)$ have the form of Eqs. (4) and (9).

Step 3: Solve the algebraic Riccati equation from Eq. (14) for $V_2(x)$ and find $\partial V_2(x)/\partial x$.

Step 4: Substitute $\partial V_2(x)/\partial x$ into Eq. (11) so that $V_3(x)$ is solved and find $\partial V_3(x)/\partial x$.

Step 5: Check the positive definiteness of the Lyapunov function. If it is not stable, then adjust the performance index and repeat step 4.

Step 6: Repeat the process until $V_{2n}(x)$ is determined.

The outlined procedure will always result in a positive definite $V(x)$. Note that for stability $V_n(x)$ has to be positive definite. This implies $V_n(x)$ must be positive definite for all even values of n. $V_n(x)$ does not have to be positive definite for odd values of n as long as $V(x)$ is positive definite.

When all Lyapunov parameters have been obtained, $V(x)$ is at least positive semidefinite, so that $\partial V(x)/\partial x$ can be calculated and the state equation becomes

$$\dot{x} = Ax + f(x) - BR^{-1}B^T \left[\frac{\partial V(x)}{\partial x}\right]^T \qquad (15)$$

which is asymptotically stable in the large. The final optimal performance index based on this method will be

$$J = \tfrac{1}{2} x^T(0) P_{1,1} x(0) + \varepsilon \qquad (16)$$

where ε depends on the initial conditions only. This term is always at least one order of magnitude smaller than the other term for the large initial conditions.

III. Results and Discussion

A. Numerical Example

The wing rock equation of motion for 80-deg slender delta wings[13] is chosen as a nonlinear example for optimal control. The equation of motion is

$$\ddot{\phi} + \varpi^2 \phi = \mu_1 \dot{\phi} + b_1 \dot{\phi}^3 + \mu_2 \phi^2 \dot{\phi} + b_2 \phi \dot{\phi}^2 + u \qquad (17)$$

Reference 13 gives

$$\varpi^2 = -c_1 a_1, \qquad \mu_1 = c_1 a_2 - c_2, \qquad b_1 = c_1 a_3$$

$$\mu_2 = c_1 a_4, \qquad b_2 = c_1 a_5$$

The values of the coefficients in these equations are shown in Table 1. Defining the state vector $x = (x_1, x_2)^T = (\phi, \dot{\phi})^T$, the system (17) can be written in state variable form as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\varpi^2 x_1 + \mu_1 x_2 + f(x) + u \end{aligned} \qquad (18)$$

where

$$f(x) = b_1 x_2^3 + \mu_2 x_1^2 x_2 + b_2 x_1 x_2^2$$

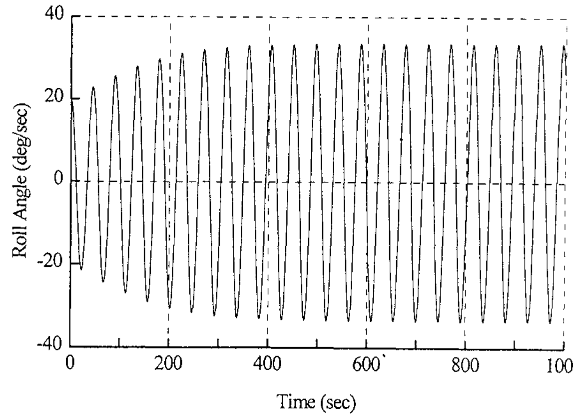

Fig. 1 Wing rock motion for α = 25 deg.

If Eq. (18) is written in the same form as Eq. (1), then

$$A = \begin{bmatrix} 0 & 1 \\ -\varpi^2 & \mu_1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (19)$$

The wing rock motion shown in Fig. 1 is indeed a limit cycle vibration. Setting

$$\dot{x} = 0$$

three equilibrium points can be found. All of these equilibrium points, however, are unstable when the nonlinear model is linearized. The method of finding optimal control for this nonlinear model starts by choosing the highest order term of $f(x)$ that results in $2n - 1 = 3$, or $n = 2$. Therefore, the performance index becomes

$$J = \frac{1}{2}\int_0^\infty \left\{ \begin{bmatrix} x \\ x^2 \end{bmatrix}^T \begin{bmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{bmatrix} \begin{bmatrix} x \\ x^2 \end{bmatrix} + u^2 \right\} \mathrm{d}t \qquad (20)$$

Note that $R = 1$. Now the Lyapunov function is assumed to be

$$V(x) = \frac{1}{2} \begin{bmatrix} x \\ x^2 \end{bmatrix}^T \begin{bmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{bmatrix} \begin{bmatrix} x \\ x^2 \end{bmatrix} \qquad (21)$$

Setting

$$Q_{1,1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and solving the algebraic Riccati equation,

$$A^T P_{1,1} + P_{1,1} A + Q_{1,1} - P_{1,1} B R^{-1} B^T P_{1,1} = 0$$

then

$$P_{1,1} = \begin{bmatrix} P_{1,1}(1,1) & P_{1,1}(1,2) \\ P_{1,1}(1,2) & P_{1,1}(2,2) \end{bmatrix}$$

where

$$P_{1,1}(1,1) = \varpi^2 \mu_1 + \frac{1}{2}\sqrt{\varpi^4 + 1}\sqrt{4\mu_1^2 - 8\varpi^2 + 8\sqrt{\varpi^4 + 1} + 4}$$

$$P_{1,1}(1,2) = -\varpi^2 + \sqrt{\varpi^4 + 1}$$

$$P_{1,1}(2,2) = \mu_1 + \frac{1}{2}\sqrt{4\mu_1^2 - 8\varpi^2 + 8\sqrt{\varpi^4 + 1} + 4}$$

Therefore,

$$V_2(x) = \tfrac{1}{2}x^T P_{1,1} x \tag{22}$$

and

$$\frac{\partial V_2(x)}{\partial x} = P_{1,1}x \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (11) results in

$$V_3(x) = \tfrac{1}{2}x^T P_{1,2}x^2 + \tfrac{1}{2}(x^2)^T P_{2,1}x \tag{24}$$

where

$$P_{2,1} = \begin{bmatrix} P_{2,1}(1,1) & P_{2,1}(1,2) \\ P_{2,1}(1,2) & P_{2,1}(2,2) \end{bmatrix}$$

Furthermore,

$$P_{2,1}(1,2) = \frac{-q_{2,1}(1,1)}{\left[-\varpi^2 - P_{1,1}(1,2)\right]}$$

$$P_{2,1}(1,1) = \tfrac{1}{3}\left\{\left[2\varpi^2 - \mu_1 + P_{1,1}(2,2) + 2P_{1,1}(1,2)\right] \times P_{2,1}(1,2) - q_{1,2}(1,2)\right\}$$

and

$$P_{2,1}(2,2) = \frac{2[P_{1,1}(2,2) - 1 - \mu_1]P_{2,1}(1,2) - q_{1,2}(1,2)}{\left[-3\varpi^2 - 3P_{1,1}(2,2)\right]}$$

Therefore,

$$\frac{\partial V_3(x)}{\partial x} = \begin{bmatrix} 3x_1^2 P_{2,1}(1,1) + x_2^2 P_{2,1}(1,2) + 2x_1x_2P_{2,1}(1,2) \\ 3x_2^2 P_{2,1}(2,2) + x_1^2 P_{2,1}(1,2) + 2x_1x_2P_{2,1}(1,2) \end{bmatrix} \tag{25}$$

To find $V_4(x)$, these results must be substituted into Eq. (13), where $n = 2$. Then,

$$V_4(x) = \tfrac{1}{2}(x^2)^T P_{2,2}x^2 \tag{26}$$

where

$$P_{2,2} = \begin{bmatrix} P_{2,2}(1,1) & P_{2,2}(1,2) \\ P_{2,2}(1,2) & P_{2,2}(2,2) \end{bmatrix}$$

and the coefficients are now given by

$$P_{2,2}(1,2) = \frac{\left[6P_{2,1}(2,2)P_{2,1}(1,2) + 4P_{2,1}^2(1,2) - q_{2,2}(1,2) - \mu_2 P_{1,1}(2,2) - b_1 P_{1,1}(1,2)\right]}{[2\mu_1 - 2P_{1,1}(2,2)]}$$

$$P_{2,2}(1,1) = P_{2,2}(1,2)P_{1,1}(1,2) + \varpi^2 P_{2,2}(1,2) + 2P_{2,1}(1,2) - 0.5\mu_2 P_{1,1}(1,2)$$

$$P_{2,2}(2,2) = \frac{\{-2P_{2,2}(1,2) + 12P_{2,1}(2,2)P_{2,2}(1,2) - b_2 P_{1,1}(2,2) - b_1 P_{1,1}(1,2)\}}{\left[-2\varpi^2 - 2P_{1,1}(1,2)\right]}$$

Therefore,

$$\frac{\partial V_4(x)}{\partial x} = \begin{bmatrix} 2x_1^3 P_{2,2}(1,1) + 2x_1x_2^2 P_{2,2}(1,2) \\ 3x_2^3 P_{2,2}(2,2) + 2x_2x_1^2 P_{2,2}(1,2) \end{bmatrix}$$

The optimal control is

$$u = -R^{-1}B^T\left(\frac{\partial V(x)}{\partial x}\right)^T = -\left[P_{1,1}(1,2)x_1 + P_{1,1}(2,2)x_2 + 3x_2^2 P_{2,1}(2,2) + (x_1^2 + 2x_1x_2)P_{2,1}(1,2) + 3x_2^3 P_{2,2}(2,2) + 2x_2x_1^2 P_{2,2}(1,2)\right]$$

Checking the positive definitenesses of the performance index and the Lyapunov function, the following equations can be found:

$$Q_{2,1}(2,2) = 3P_{2,1}(2,2)P_{1,1}(2,2) - P_{2,1}(1,2) - 3\mu_1 P_{2,1}(2,2)$$

$$Q_{2,2}(1,1) = 2P_{2,1}(1,2)^2$$

These equations ensure that the performance index and the Lyapunov function are positive definite and HJB equation is satisfied.

The following results are found using the coefficients for $\alpha = 25$ deg. The performance index is set to

$$J = \frac{1}{2}\int_0^\infty \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}^T \times \begin{bmatrix} 1 & 0 & 1/64 & 0 \\ 0 & 1 & 0 & -0.05434 \\ 1/64 & 0 & 4.88\cdot 10^{-4} & 0 \\ 0 & -0.05434 & 0 & 3.6758 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix} + u^2 \right\} \mathrm{d}t$$

and the Lyapunov function is determined to be

$$V(x) = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}^T \times \begin{bmatrix} 1.7211 & 0.98 & 0.019376 & 0.01562 \\ 0.98 & 1.7285 & 0.01562 & -0.0075 \\ 0.019376 & 0.01562 & 0.3551 & 0.08388 \\ 0.01562 & -0.0075 & 0.08388 & 0.547 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \\ x_2^2 \end{bmatrix}$$

The results of optimal control of the present wing rock equation of motion are shown in Figs. 2 and 3. The optimal control as shown in Fig. 4 is

$$u = -\left[0.9811x_1 + 1.7285x_2 - 0.0225x_2^2 + 0.01562 \times \left(x_1^2 + 2x_1x_2\right) + 1.094x_2^3 + 0.1678x_2x_1^2\right]$$

Note that the performance index and the Lyapunov function are positive definite.

The complete set of results for all three angles of attack of Ref. 13 are shown in Tables 2 and 3. Again, all performance indices and the Lyapunov functions shown in Tables 2 and 3 are positive definite.

B. Results and Comparisons

Consider control architecture of the nonlinear system with linear feedback and nonlinear feedback shown in Figs. 5 and 6. The linear feedback model is based on the nonlinear system linearized about the equilibrium point (0, 0). In fact, the optimal feedback control of this linear model is the same as the second order results of the nonlinear

Table 2 Coefficients of the closed-loop Lyapunov function

α	$P_{1,1}(1,1)$	$P_{1,1}(1,2)$	$P_{1,1}(2,2)$	$P_{2,1}(1,1)$	$P_{2,1}(1,2)$	$P_{2,1}(2,2)$	$P_{2,2}(1,1)$	$P_{2,2}(1,2)$	$P_{2,2}(2,2)$
21.5	1.7238	0.9852	1.7277	0.0194	0.0156	−0.0075	0.1102	−0.0085	0.0745
22.5	1.7229	0.9836	1.7285	0.0194	0.0156	−0.0075	0.1801	0.0193	0.2106
25	1.7211	0.9801	1.7285	0.0194	0.0156	−0.0075	0.3551	0.0839	0.547

Table 3 Coefficients of the performance index

α	$q_{1,1}(1,1)$	$q_{1,1}(1,2)$	$q_{1,1}(2,2)$	$q_{2,1}(1,1)$	$q_{2,1}(1,2)$	$q_{2,1}(2,2)$	$q_{2,2}(1,1)$	$q_{2,2}(1,2)$	$q_{2,2}(2,2)$
21.5	1	0	1	1/64	0	−0.0546	0.0005	0	0.4569
22.5	1	0	1	1/64	0	−0.0545	0.0005	0	1.3825
25	1	0	1	1/64	0	−0.0543	0.0005	0	3.6758

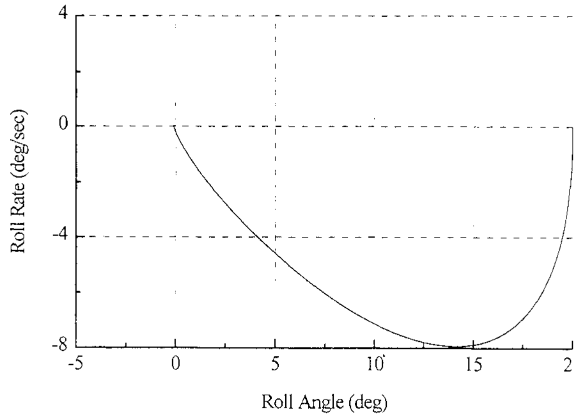

Fig. 2 Phase plane plot of the wing rock motion with optimal nonlinear state feedback.

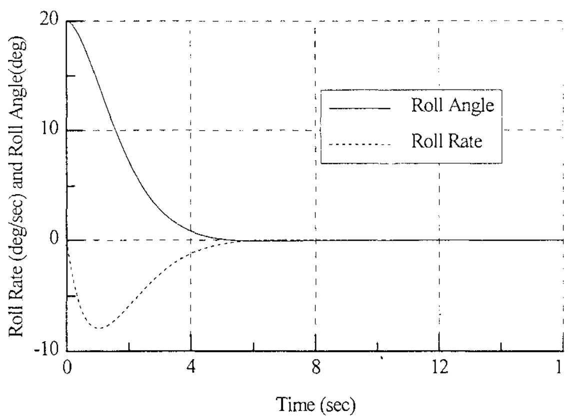

Fig. 3 State response of optimal nonlinear control.

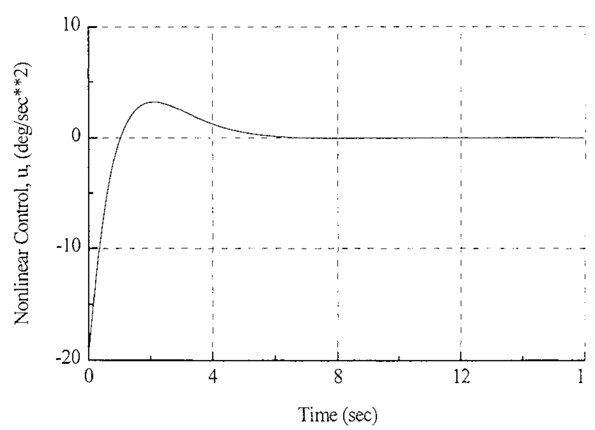

Fig. 4 Time history of the optimal feedback controller.

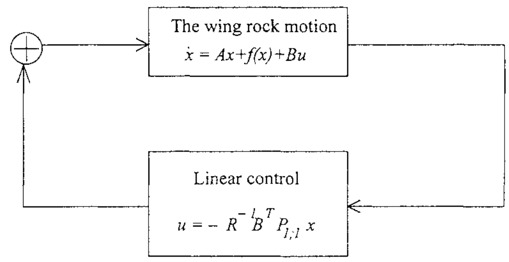

Fig. 5 Control architecture for the linear optimal feedback case.

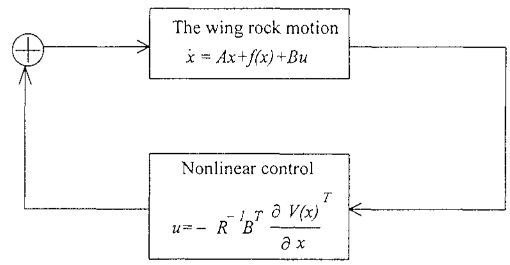

Fig. 6 Control architecture for the nonlinear optimal feedback case.

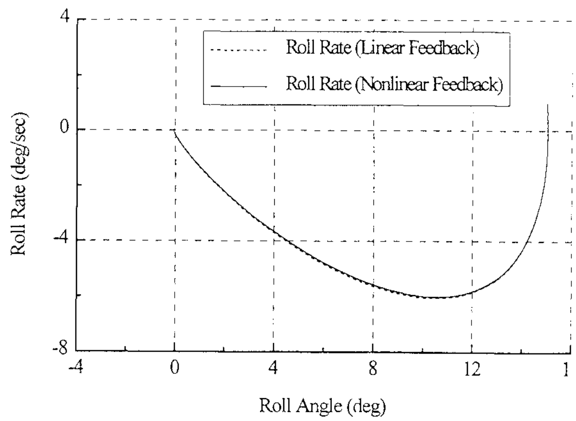

Fig. 7 Phase plane plot of the optimal wing rock motion in the small condition.

optimal feedback control, i.e., $u = -R^{-1}B^T[\partial V_2(x)/\partial x]^T$. Results for the control of the wing rock motion based on control architectures of Figs. 5 and 6 are shown in Figs. 7 and 8. Figure 7 shows that the nonlinear feedback control and linear feedback control are not very different when the system is in the small. When in large initial conditions, however, significant differences between the nonlinear feedback and linear feedback appear. Figure 9 shows that when the initial conditions are set to 80 deg, 200 deg/s, the nonlinear feedback control system can still stabilize and optimize the system. The linear feedback model diverges very quickly, however, reaching

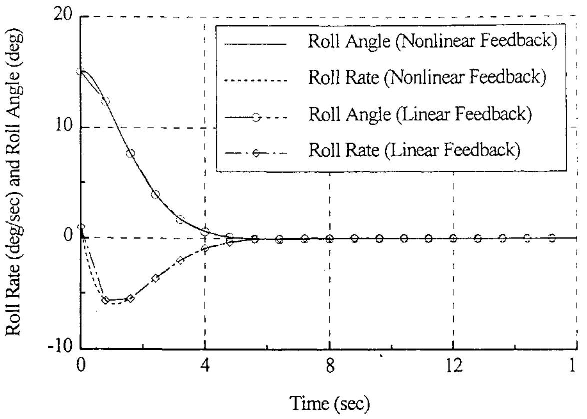

Fig. 8 State response to both linear and nonlinear feedback for small initial conditions.

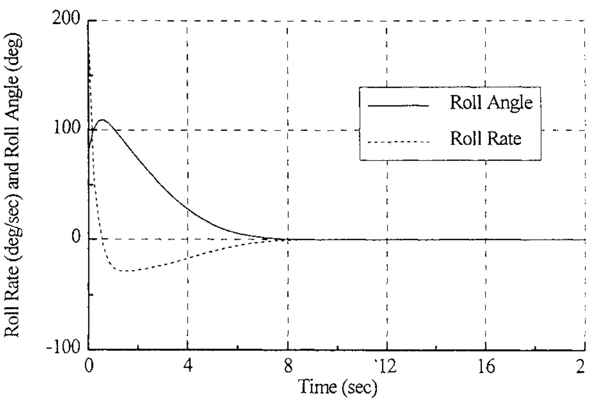

Fig. 9 State response to nonlinear state feedback for large initial conditions.

$(x_1, x_2)^T = (232, 7960)^T$ in 0.37 s. This implies that the nonlinear feedback control is indeed asymptotically stable in the large.

C. Discussion

The problem of optimal control of nonlinear systems has been studied for more than 30 years. Research in this area, however, has emphasized the solution of the HJB equation, ignoring the fact that the performance index and the Lyapunov function must be at least positive semidefinite. From the second method of the Lyapunov theory,[14] when

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = \frac{\partial V(x)}{\partial x}(Ax + f(x) + Bu)$$
$$= -q(x) - \tfrac{1}{2}u^T Ru$$

the remaining conditions, $q(x) \geq 0$ and $V(x) \geq 0$, must also be satisfied. This guarantees that the optimal control of a nonlinear system is asymptotically stable in the large.

Here, since the performance index is assumed as Eq. (3), the interpolated term of $q(x)$ is

$$Q = \begin{bmatrix} Q_{1,1} & Q_{2,1} & \cdots & \cdots & Q_{n,1} \\ Q_{2,1} & Q_{2,2} & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ Q_{n,1} & \cdots & \cdots & \cdots & Q_{n,n} \end{bmatrix}$$

which is positive definite. Furthermore, the closed-loop Lyapunov function is assumed to have the same form as the performance index. Therefore, the interpolated term of $V(x)$,

$$P = \begin{bmatrix} P_{1,1} & P_{2,1} & \cdots & P_{n,1} \\ P_{2,1} & P_{2,2} & & \vdots \\ \vdots & & \ddots & \vdots \\ P_{n,1} & \cdots & \cdots & P_{n,n} \end{bmatrix}$$

is easily seen to be also positive definite. The control computed from the partial derivative of the closed-loop Lyapunov function with respect to the state variable allows the system to be stable in the large. This is because the HJB equations are satisfied by the given positive definite performance index and closed-loop Lyapunov function.

IV. Conclusion

A new method was presented for solving nonlinear optimal control problems. The control of a nonlinear system was guaranteed to be stable in the large by satisfying the second method of the Lyapunov theory. It was determined that the Lyapunov function can be made positive definite by proper choice of a positive definite performance index. A simple way of defining a positive definite performance index and Lyapunov function was shown to be using the same matrix form for both. These state variable expansion series matrix forms allow the interpolated constant matrices of both functions to be easily checked for their positive definitenesses. The optimal and asymptotically stable system using nonlinear state feedback control was found, when both positive definite functions were determined. A wing rock model was used to illustrate that the nonlinear feedback control system was asymptotically stable in the large.

Even though the major emphasis was placed on second-order systems in this paper, the procedure outlined here is applicable to any system of finite order.

References

[1]Garrard, W. L., "Suboptimal Feedback Control for Nonlinear Systems," *Automatica*, Vol. 8, Aug. 1972, pp. 219–221.

[2]Baldwin, J. F., and Williams, J. H. S., "The Use of a Method of Perturbations in the Synthesis of Closed-Loop Optimal Control Law for Nonlinear Systems," *Automatica*, Vol. 5, May 1969, pp. 357–367.

[3]Garrard, W. L., McClamroch, N. H., and Clark, L. G., "An Approach to Suboptimal Feedback Control of Nonlinear System," *International Journal of Control*, Vol. 5, May 1967, pp. 425–435.

[4]Al'breht, E. G., "On the Optimal Stabilization of Nonlinear Systems," *Journal of Applied Mathematics and Mechanics*, Vol. 25, 1961, pp. 1254–1267.

[5]Zhang, Y.-L., Gao, J.-C., and Zhou, C.-H., "Optimal Regulator of Nonlinear System," *International Journal of Control*, Vol. 50, No. 3, 1989, pp. 993–1000.

[6]Shaw, L., "Nonlinear Control of Linear Multivariable Systems via State-Dependent Feedback Gains," *IEEE Transactions on Automatic Control*, AC-24, No. 1, 1979, pp. 108–112.

[7]Frankena, J. F., and Sivan, R., "A Non-linear Optimal Control Law for Linear Systems," *International Journal of Control*, Vol. 30, No. 1, 1979, pp. 159–178.

[8]Sandor, J., and Williamson, D., "Design of Nonlinear Regulators for Linear Plants," *IEEE Transactions on Automatic Control*, AC-22, No. 1, 1977, pp. 47–50.

[9]Garrard, W. L., and Jordan, J. M., "Design of Nonlinear Automatic Flight Control Systems," *Automatica*, Vol. 13, No. 5, 1977, pp. 497–505.

[10]Goh, C. J., "On the Nonlinear Optimal Regulator Problem," *Automatica*, Vol. 29, No. 3, 1993, pp. 751–756.

[11]Bernstein, D. S., "Nonquadratic Cost and Nonlinear Feedback Control," *Proceedings of the 1991 American Control Conference*, pp. 533–538.

[12]Leeper, J. L., and Mulholland, R. J., "Optimal Control of Nonlinear Single-Input Systems," *IEEE Transactions on Automatic Control*, AC-17, No. 3, 1972, pp. 401, 402.

[13]Nayfeh, A. H., Elzebda, J. M., and Mook, D. T., "Analytical Study of the Subsonic Wing-Rock Phenomenon for Slender Delta Wing," *Journal of Aircraft*, Vol. 26, No. 9, 1989, pp. 805–809.

[14]Siljak, D. D., *Nonlinear System the Parameter Analysis and Design*, Wiley, New York, 1969, pp. 316–320.